

Conductance Distributions in Random Resistor Networks. Self-Averaging and Disorder Lengths

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The self-averaging properties of the conductance g are explored in random resistor networks (RRN) with a broad distribution of bond strengths $P(g) \sim g^{\nu-1}$. The RRN problem is cast in terms of simple combinations of random variables on hierarchical lattices. Distributions of equivalent conductances are estimated numerically on hierarchical lattices as a function of size L and the distribution tail strength parameter μ . For networks above the percolation threshold, convergence to a Gaussian basin is always the case, except in the limit $\mu \rightarrow 0$. A *disorder length* ξ_D is identified, beyond which the system is effectively homogeneous. This length scale diverges as $\xi_D \sim |\mu|^{-\nu}$ (ν is the regular percolation correlation length exponent) when the microscopic distribution of conductors is exponentially wide ($\mu \rightarrow 0$). This implies that exactly the same critical behavior can be induced by geometrical disorder and by strong bond disorder with the bond occupation probability $p \leftrightarrow \mu$. We find that only lattices at the percolation threshold have renormalized probability distributions in a *Levy-like* basin. At the percolation threshold the disorder length diverges at a critical tail strength μ_c as $|\mu - \mu_c|^{-z}$ with $z \sim 3.2 \pm 0.1$, a new exponent. *Critical path analysis* is used in a generalized form to give the macroscopic conductance in the case of lattices above p_c .

KEY WORDS: Resistor networks; hierarchical lattices; disorder; probability distributions.

INTRODUCTION

Scaling properties of transport quantities in random media have attracted great interest for some time. Electric conduction on random networks, in particular, is of great relevance because it may be regarded as a simple qualitative archetype of other, more complex and less well-understood problems in disordered systems. Such physical systems include the classical

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problem of flow in porous media,⁽¹⁾ diffusion in a random environment⁽²⁾ and even the quantum problem of hopping conduction in dirty semiconductors⁽³⁾ or metal insulator composites,⁽⁴⁾ among others. These systems can be modeled on a random network where individual bonds represent a 'resistor' whose properties depend on the quantity which flows through it. In the case of flow in porous media this resistor represents a microscopic permeability or *permeance* (in analogy with *conductance* to indicate that it may not be an intensive quantity). For hopping conduction, on the other hand, the resistor involves the computation of the overlap integral between impurities, and quantum effects are taken into account.⁽³⁾ The physics involved in the computation of the resistors determines an associated probability distribution function (PDF) for values on the equivalent resistor network. The resulting distribution for hopping conduction is exponentially wide [$r = r_0 \exp(-\varepsilon)$ with ε , say, uniformly distributed], while in the case of flow through porous media in the capillary regime it has been argued that the distribution has a power law component and a large-permeability cutoff.⁽⁵⁾ In the same fashion, continuum percolation results in a random network with a power law distribution of resistance $P(r) \sim 1/r^{\mu+1}$.⁽⁶⁾ In addition to the *statistical* disorder discussed above, one can also introduce *geometrical* disorder by allowing voids in the network with a certain probability. Networks with a broad distribution of bond values at the percolation threshold have been extensively studied numerically and by field-theoretic techniques.⁽⁷⁻⁹⁾

Once the distribution of the elementary resistors is set, as discussed above, one is interested in computing the macroscopic properties of such networks. How does the probability distribution of the equivalent conductance 'renormalize' as the network is rescaled? What is the effective support for transport, given the elementary distribution of resistors? In brief, how do the random variables *combine* and thus 'interact' with the microscopic probability distribution (MPD) to give a macroscopic result. In this work we explore the scaling properties of conduction on *hierarchical networks* where the problem can be simply and explicitly stated in terms of a nonlinear combination of random variables. This approach is very attractive since it dwells on the problem of stable limit distributions for combinations of random variables (other than additive) for which there are few rigorous results. This approach has been adopted recently by Derrida and Griffiths,⁽¹⁰⁾ Halpin-Healy,⁽¹¹⁾ and Roux *et al.*⁽¹²⁾ in the context of directed polymers in a random medium and by Bouchaud, Le Doussal, and Georges in a series of papers reviewed in ref. 2 in the context of anomalous diffusion.

The paper is organized as follows: In Section 1 we discuss hierarchical lattices and how resistor network composition laws are easily implemented.

We then show the results for the evolution of the PDF as a function of system size for hierarchical networks at and above the percolation threshold. The existence of two basins of attraction is obtained as a function of the MPD tail strength parameter. In Section 2 we discuss the existence of a disorder length ξ_D below which strong heterogeneities persist, and derive the conductance at this scale by *critical path analysis* introduced by Ambegaokar *et al.* (AHL).^(3,13) This length scale determines the crossover distance over which the system self-averages. We also show the divergence of ξ_D as one approaches $P(g) \sim 1/g$ and determine the associated critical exponent numerically and using simple analytical arguments. We find that this exponent is exactly that of regular percolation, and establish a mapping between the probability p of regular percolation and the tail strength parameter μ . In Section 3 we point out the parallelism between Levy limit theorems and limit distributions found in random resistor networks. Analogies to other disordered systems are also discussed.

1. HIERARCHICAL LATTICES

Hierarchical lattices are self-similar structures generated by an iterative procedure that produces the lattice at order $m + 1$ from order m , by substituting every bond by a chosen motif (see Fig. 1). With this procedure one obtains lattices of different effective dimensions d_e and connectivities, which have proven useful in exposing the qualitative behavior of low-dimensional systems.⁽¹⁴⁾ Figure 1 shows the so-called ‘Berker lattice’ of effective dimension $d_e = 2$ and the Arcangelis *et al.* (ARC) lattice⁽¹⁵⁾ frequently used as a model for percolating backbones ($d_e = \log 4 / \log 3$).

Bonds on the hierarchical lattices are assigned conductances g , independently chosen from the probability distribution

$$P(g) = |\mu| g^{\mu-1} \quad \text{with} \quad \begin{cases} 0 \leq g \leq 1 & \text{if } \mu > 0 \\ g > 1 & \text{if } \mu < 0 \end{cases} \quad (1)$$

The ranges assigned to the conductance g secure that the distribution is normalized for the given values of the tail strength μ . Various limiting behaviors can be achieved in Eq. (1) by varying μ , namely: (a) $\mu \gg 1$ corresponds to $P(g) \rightarrow \delta(g - 1)$, (b) $\mu = 1$ to a flat-top distribution between $0 < g < 1$, (c) $0 < \mu < 1$ to continuum percolation, and (d) $\mu < 0$ to algebraic tails for $g \rightarrow \infty$. In addition one can study the limit (e) $\mu \rightarrow 0^\pm$, which corresponds to the relevant distribution for dirty semiconductors in the hopping regime.⁽¹³⁾

Some background and notation for transport and percolation exponents are in line: For infinite systems near p_c above the threshold, the

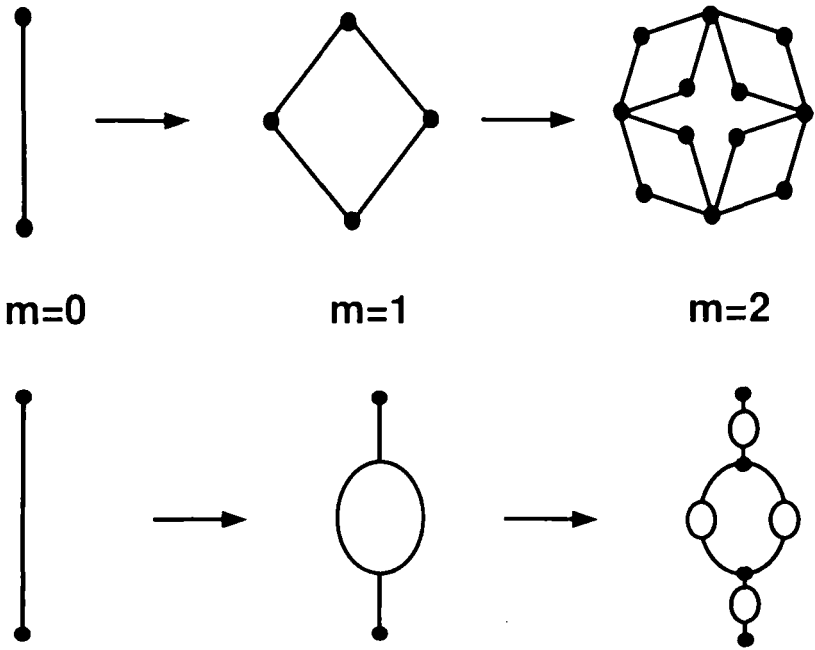


Fig. 1. Recursive construction of a hierarchical lattice: Each bond at a given generation m is converted into a chosen motif. Top: The Berker lattice $d_c = 2$; bottom: the ARC lattice of $d_c = \log 4 / \log 3$.

conductivity σ behaves as $\sigma \sim (p - p_c)^t$, where t is the conductivity exponent. In the percolation problem a characteristic length $\xi \sim (p - p_c)^{-\nu}$ of geometrical origin emerges. The correlation length exponent ν is defined by the previous expression. Below this length scale the system is strongly inhomogeneous and has a fractal structure, and thus intensive quantities such as the conductivity scale with the length L in a nontrivial manner. Beyond this scale the system becomes effectively homogeneous, and intensive quantities take their macroscopic value (the macroscopic average conductance $\langle G \rangle = \sigma L^{d-2}$ with σ independent of L). It is an accepted theoretical result⁽⁷⁻⁹⁾ that when the PDF of conductors corresponds to the continuum percolation distribution and the underlying lattice is at p_c , one finds a regime where universal *lattice* exponents are found, while values of μ below a certain threshold μ_c yield tail-strength-dependent exponents. This behavior is summarized by the expression $t = \max\{t_0, t(\mu)\}$, where $t_0 = (d - 2)\nu + \phi_0$ and $t(\mu) = (d - 2)\nu + 1/\mu$.

Hierarchical lattices of length L ranging from 2 to 16384 with random microscopic conductances obtained from Eq. (1), are generated for selected

values of μ . The equivalent conductance G of each lattice is then calculated exactly. More than 10,000 realizations of randomness are generated to produce a sample from which a renormalized distribution of equivalent conductances can be estimated. Our results are the following:

1. For lattices above the percolation threshold, we find a rapid convergence of the MPD to a sharp Gaussian for all $\mu \neq 0$, as shown in Fig. 2. The conductance is then a *self-averaging* quantity in the sense that, as the network increases in size, the conductance approaches a limit value with diminishing fluctuations. For the conductance one finds that $\langle G \rangle \sim G_{\text{typ}}$ for large enough L , indicative of a peaked distribution (G_{typ} stands for most probable conductance).

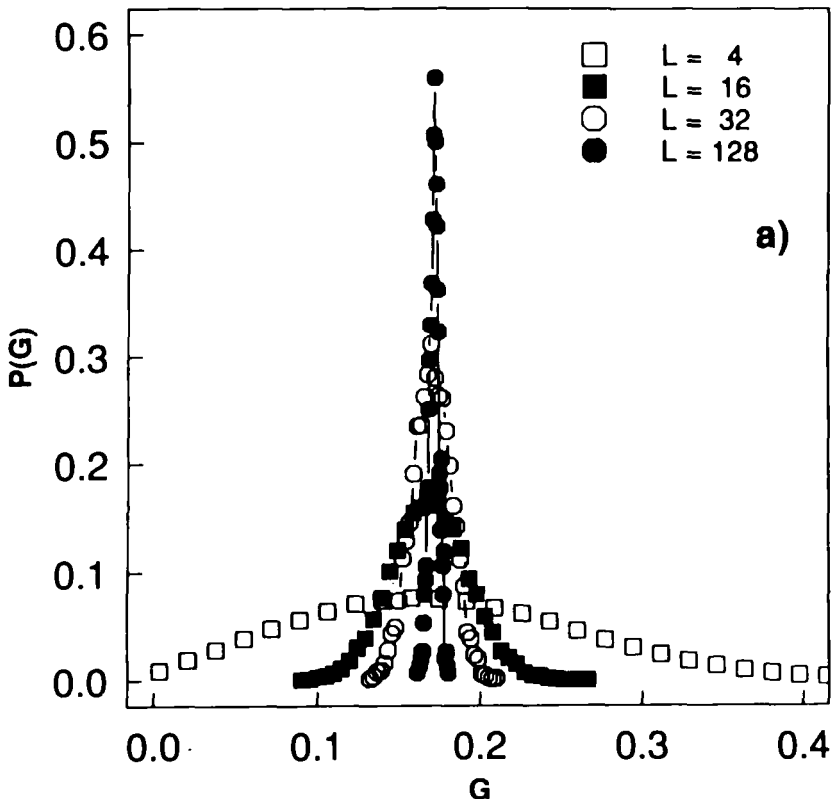


Fig. 2. Convergence of a broad microscopic distribution toward a Gaussian as L is increased. (a) The case of the Berker lattice of $d_c=2$ and $\mu=0.4$; (b) (next page) the Berker lattice for $d_c=3$ and $\mu=0.3$. The size of the lattice used is (squares) $L=16$, (filled circles) $L=256$, (triangles) $L=16,384$. The inset shows a fit to a Gaussian distribution.

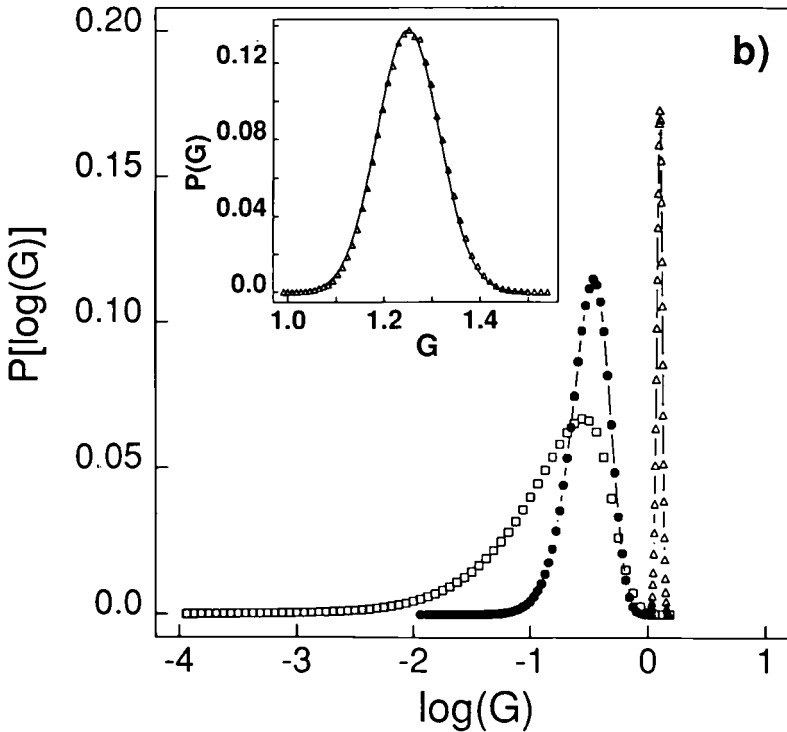


Fig. 2. (Continued)

Figure 3 shows the behavior of the conductance as a function of the length L . Two well-defined regions are apparent; for small networks the conductance drops sharply, while beyond a given disorder length ξ_D , classical ($G \sim L^{d-2}$) behavior is recovered. The curves for different values of μ can be collapsed by scaling L by ξ_D and the conductance G by G_{ξ_D} (the conductance at the disorder length). The result is two universal curves for $\mu > 0$ and $\mu < 0$. In the limit $\mu \rightarrow 0$, the collapse indicates that ξ_D diverges as a power of the tail strength parameter μ , i.e., $\xi_D = A_{\pm} |\mu|^{-z}$, as depicted in Fig. 3. The ratio of the amplitudes is $A_+/A_- = 0.6$ and the exponent $z = 1.6 \pm 0.1$. The subscripts on the prefactor indicate the approach from either $\mu > 0$ or $\mu < 0$. For lengths larger than ξ_D , the conductivity reaches its macroscopic value (given $\xi < \xi_D$). One can then write (in d dimensions)

$$\langle G \rangle = \langle G_{\xi_D} \rangle (L/\xi_D)^{d-2} = \left\langle \frac{G_{\xi_D}}{\xi_D^{d-2}} \right\rangle L^{d-2} = \sigma L^{d-2} \quad (2)$$

which defines the macroscopic conductivity $\sigma = \langle G_{\xi_D}/\xi_D^{d-2} \rangle$. One can go

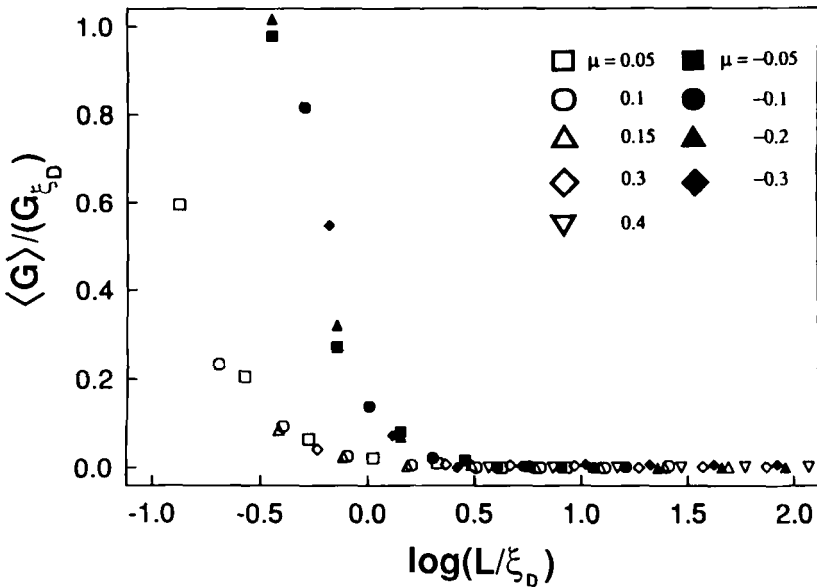


Fig. 3. The conductance versus the scaling variable L/ξ_D for a lattice of $d_c=2$, above the threshold. A crossover length ξ_D and a characteristic conductance G_{ξ_D} are identified by collapse of different μ values. The length ξ_D diverges when $\mu \rightarrow 0$ as $A_{\pm} |\mu|^{-1.6 \pm 0.1}$, where $A_+/A_- = 0.6$.

further and derive the form of $\langle G_{\xi_D} \rangle$ using critical path analysis, valid for ξ_D sufficiently large.^(13,16) This is done in the next section. For $d_c=3$ collapse is achieved by scaling the vertical axis by $G_{\xi_D} = (1 - p_c)^{v+1/\mu} \mu^v$, a scaling variable which will be discussed in the next section. In Fig. 4 we show the effective support for conduction on a hierarchical lattice of $d_c=2$ conveniently drawn for the purpose of illustration (top of figure). We depict only the bonds supporting 99% of the current on the hierarchical network as a function of μ . The disorder length ξ_D derived above is related to the vertical length of the voids. As disorder increases ($\mu \rightarrow 0$) the current is essentially carried by a percolation backbone.

In summary, no matter how broad the MPD (except for $\mu=0$), the limiting behavior for $P(g)$ on hierarchical networks of $d_c=2, 3$ converges to a sharp Gaussian. Geometrical disorder produced when $p_c < p < 1$ seems not to affect the limit behavior of $P(g)$. A disorder length ξ_D is found for the convergence to the ‘Gaussian’ basin, which defines the length beyond which fluctuations in the conductance decrease around an average value $\langle G \rangle = G_{typ}$.

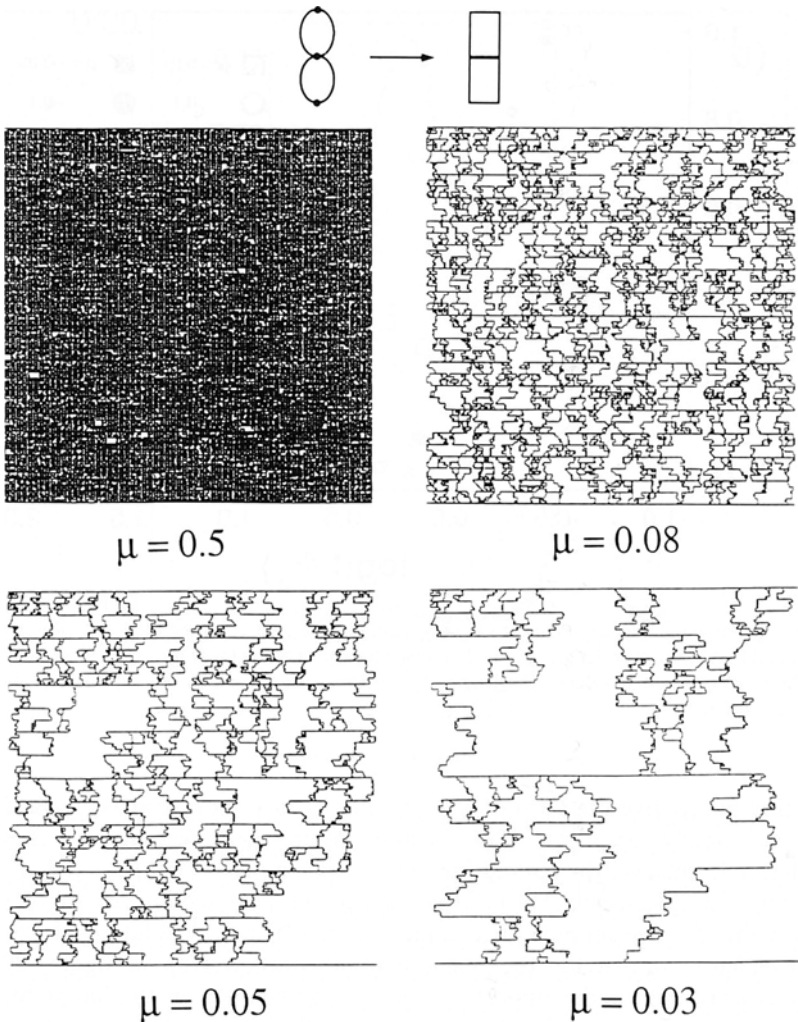


Fig. 4. Lattice of bonds carrying 99% of the current as a function of μ . The hierarchical lattice generator is conveniently drawn as shown in the figure. Only the lengths along the vertical axis in the figure are meaningful.

2. For networks at the threshold, two regimes occur as found by Machta *et al.* (MGM).⁽¹⁷⁾ For $\mu > \mu_c \sim 0.75$ the system is self-averaging, so that a peaked distribution is obtained for large L . In this regime universal lattice exponents are recovered. We note that the limit distribution for resistors, although highly peaked, preserves a degree of skewness (as measured by the third cumulant) as far as we can determine. This means

that the limit PDF may not approach a Gaussian behavior as found for lattices above the threshold. On the other hand, for $\mu < \mu_c$ the MPD tails are preserved (see Fig. 5) and nonuniversal conductivity exponents are obtained. For the ARC lattice we find a disorder length ξ_D by scaling the conductance of the network by L^α with $\alpha = \ln(2/5)/\ln 3$. This scale factor corresponds to the scaling of the conductance for the ARC network depicted in Fig. 2, with no randomness present. Figure 6 shows evidence of a disorder length by collapsing curves for different values of μ . The horizontal axis is scaled by $\xi_D = A_+(\mu - \mu_c)^{-z'}$. By collapsing the curves (see Fig. 6) we find $\mu_c = 0.75 \pm 0.05$ and $z' = 3.2 \pm 0.1$. The value of μ_c is close to that reported by MGM on the basis of numerical estimates. The exponent z' , on the other hand, is a *new exponent*. Presumably this new length scale determines a substructure of the ARC lattice supporting most

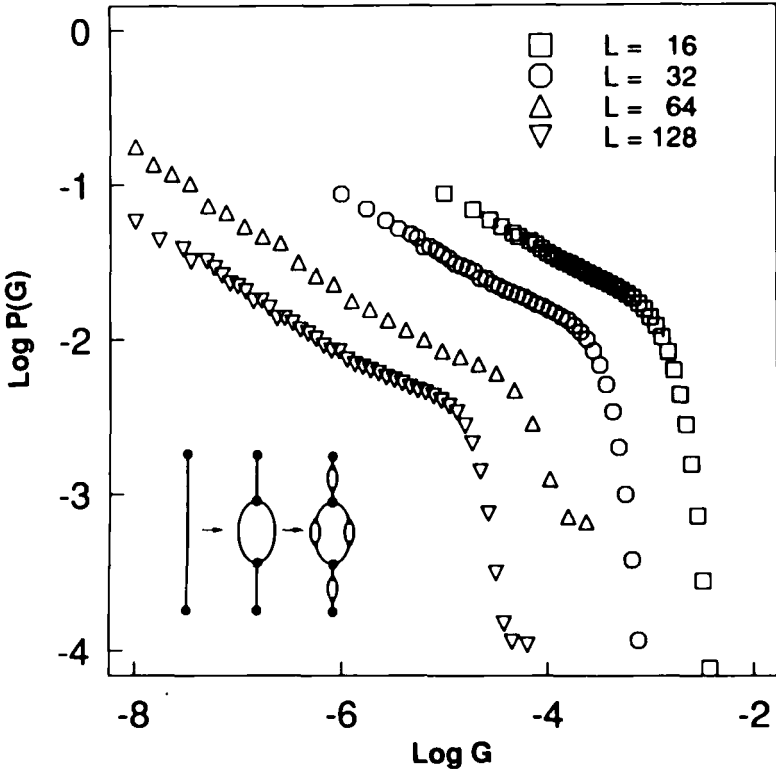


Fig. 5. Limit distribution for a percolating geometry (ARC lattice) and $\mu = 0.5 < \mu_c$. The resulting distribution shifts toward low G values, while power law tails are preserved. Log-log scales are used so power law tails are manifest.

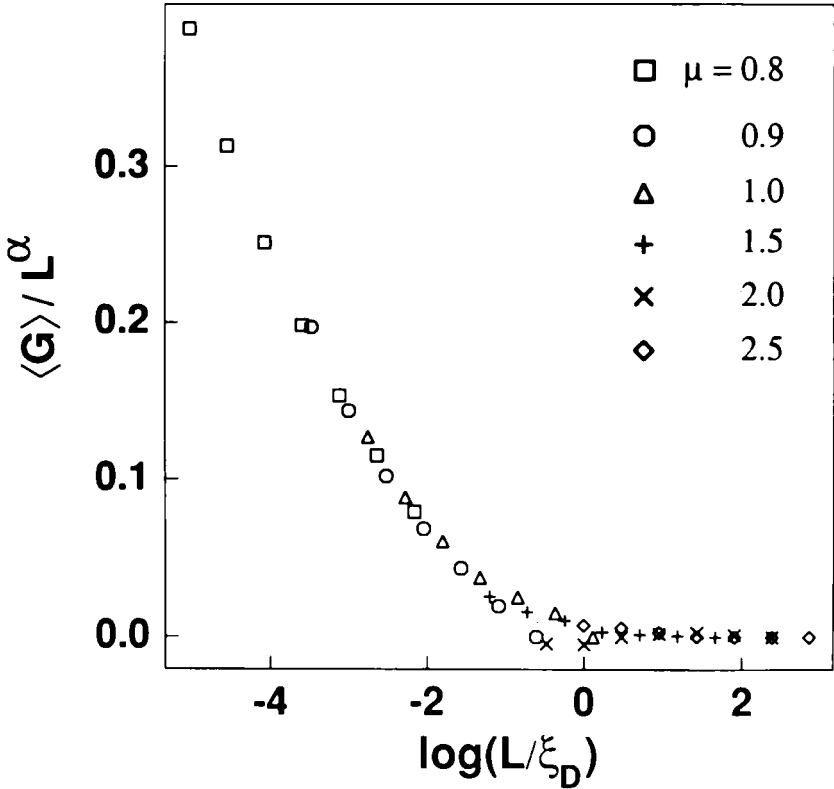


Fig. 6. Conductance versus system length for the ARC lattice with $\mu > \mu_c$. The collapse is achieved using $\xi_D = A_+(\mu - \mu_c)^{-z'}$, where $z' = 3.2 \pm 0.1$, $A_+ \sim 500$. On the vertical axis $\alpha = \ln(2/5)/\ln 3$.

of the current. The prefactor $A_+ \sim 500$ is large in comparison to the ones obtained above the threshold and is indicative of long crossover effects, well recognized in the literature.⁽¹⁸⁾ Regarding the scaling of the average conductance at the disorder length (G_{ξ_D}), we find a very weak but clear μ dependence as $G_{\xi_D} \sim B \log \mu$.

Below μ_c , ξ_D remains infinite and MDP tails are preserved through the scales as reported before.^(17,19) We call this behavior *Levy-like* for its similarities to the case of addition of random variables, when the tail strength is such that the first and second moments diverge. The additive limit theorem is obviously equivalent to the resistor network in one dimension (in terms of resistors).

2. DISORDER LENGTH

In the last section the existence of a disorder length scale ξ_D for homogenization was shown numerically. In the present section we discuss the origin of this length scale for networks above the threshold, starting from the theory of Ambegaokar *et al.*⁽¹³⁾ This theory was originally applied to transport processes that involve quantum mechanical tunneling or thermal activation over a barrier, where the barrier distribution is itself broad. The relevant distribution for these systems is of the form

$$P(g) = \frac{1}{\lambda g} \quad \text{with} \quad 1 < g < e^\lambda \quad (3)$$

where the range for g is restricted so that the distribution is normalizable. This PDF is so broad as $\lambda \rightarrow \infty$ that conduction is restricted to a small subset of the whole supporting structure. AHL conjectured that the relevant subnetwork would be dominated by the bottleneck resistor which first establishes a conducting path by the following procedure: remove all conductors recalling their location of origin, then put them back in order of decreasing conductance. The bond laid at the point of percolation will determine the macroscopic conductance. This process is formally expressed by the *percolation condition*

$$\int_{g_c}^{\infty} P(g) dg = p_c \quad (4)$$

where g_c is the governing conductance. This argument is applicable always when λ is large so that $P(g)$ in Eq. (3) has strong enough tails. In the following we shall establish, by plausible arguments, the dependence of the disorder length ξ_D on the tail strength parameter μ . Arguments will pertain mostly to the results above the percolation threshold described in Section 1.

By definition the correlation length is

$$\xi = l_0(p - p_c)^{-\nu} = l_0 \lambda^\nu [\ln(g_c/g)]^{-\nu} \quad (5)$$

where we have used Eq. (3) and the percolation condition. The prefactor $l_0 \lambda^\nu$ defines a length scale which we identify with the disorder length ξ_D . The argument builds on a relation that suggests we use the critical path analysis for more general distributions,⁽¹⁶⁾ namely

$$P(g_c) g_c = \lambda^{-1} \ll 1 \quad (6)$$

which simply states that λ should be large. Applying the percolation

condition for the power law distributions in Eq. (1), one arrives at the following dependence for g_c :

$$g_c = p_c^{1/\mu} \quad \text{for } \mu < 0$$

Using the condition in Eq. (6) yields

$$p_c g_c = |\mu| g_c^\mu = |\mu| p_c = \lambda^{-1} \quad \text{for } \mu < 0$$

By Eq. (5) one can establish that $\lambda = (\xi_D/l_0)^{1/\nu}$, so for $\mu < 0$

$$|\mu| p_c = (\xi_D/l_0)^{-1/\nu}$$

from which

$$\xi_D^{(-)} = \frac{l_0}{p_c^\nu} |\mu|^{-\nu} \tag{7}$$

Analogously for $\mu > 0$ one gets

$$\xi_D^{(+)} = \frac{l_0}{(1 - p_c)^\nu} |\mu|^{-\nu}$$

From these expressions we can also derive the ratio of the amplitude (not universal), which comes to be

$$A_+/A_- = [(1 - p_c)/p_c]^{-\nu} \tag{8}$$

Taking the values for p_c and ν , which can be computed exactly on these lattices, we get $A_+/A_- = 0.46$, in reasonable agreement with the value 0.6 found numerically. This simple argument leads to the conclusion that $z = \nu$, the percolation correlation length exponent, and the correspondence $\mu \leftrightarrow p$. One can also arrive at the same conclusion by a simpler derivation as follows: From $\xi = l_0(p - p_c)^{-\nu}$ and using Eqs. (1)–(4), one obtains $\xi = l_0(g^\mu - g_c^\mu)^{-\nu}$. As we are interested in the $\mu \rightarrow 0$ limit, we expand g^μ in powers of μ and find to first order

$$\xi = l_0 \left| \frac{\ln g}{\ln g_c} \right|^{-\nu} |\mu|^{-\nu}$$

from which the same length scale ξ_D is identified. One can alternatively state the relation between the AHL distribution springing from $r = r_0 \exp(\lambda \varepsilon)$

and the distributions given by Eq. (1) by choosing ε from the appropriate distribution $D(\varepsilon)$.⁽¹⁶⁾

One can readily explain the scaling variable for the conductance in Fig. 3 by invoking a relation first suggested by Ambegaokar *et al.*⁽²⁰⁾ and later discussed in general dimensions by Tyč and Halperin,⁽¹⁶⁾

$$\langle G_{\xi_D} \rangle = C g_c [g_c P(g_c)]^{(d-2)\nu} \tag{9}$$

where g_c is the critical conductance and as $\lambda \rightarrow \infty$, $P(g)$ is a broad distribution and ν is the correlation length exponent for ordinary percolation. From Eq. (6) the conductance $\langle G_{\xi_D} \rangle$ depends on λ . In $d=2$ no λ dependence is present and AHL arguments have no corrections. As a matter of fact, on square lattices it can be shown by duality arguments that $C = 1$ in Eq. (9) and AHL arguments are exact. For the general power law distribution in Eq. (1) one gets

$$\langle G_{\xi_D} \rangle = \begin{cases} C_1(1 - p_c)^{1/\mu} & \text{for } d=2 \\ C_2(1 - p_c)^{\nu + 1/\mu} \mu^\nu & \text{for } d=3 \end{cases} \tag{10}$$

valid for $\mu > 0$. Similar formulas (with $1 - p_c \rightarrow p_c$) hold for $\mu < 0$. In general dimensions we expect

$$\langle G_{\xi_D} \rangle = \begin{cases} (1 - p_c)^t F_1(L/\mu^{-\nu}) & \text{for } \mu > 0 \\ p_c^t F_2(L/\mu^{-\nu}) & \text{for } \mu < 0 \end{cases} \tag{11}$$

with $t = (d - 2)\nu + 1/\mu$ the nonuniversal conductivity exponent, and F_1 and F_2 universal functions. This is precisely the behavior verified by collapsing data in two and three dimensions for $\mu \rightarrow 0$. When μ is larger, these predictions are expected to deteriorate as the volume within ξ_D gets smaller and the AHL percolation arguments include further corrections.

The previous discussion has only involved networks above the percolation threshold. We have no similar understanding for phenomena occurring at the threshold. Most likely, the behavior observed in this range will follow from a renormalization group treatment similar to that of Lubensky and Tremblay.⁽⁸⁾ This treatment correctly shows (with corrections due to Machta⁽⁹⁾) that there is a crossover from normal conductivity exponents to nonuniversal ones at μ_c for lattices at the threshold. What transpires from the numerical results is that the percolating lattice with strong disorder for $\mu > \mu_c$ self-averages after a length scale ξ_D . Above this length scale the system behaves as if no disorder were present, i.e., the

length dependence of the conductivity is purely of geometrical origin. As μ approaches μ_c , this length diverges, giving way to a non-self-averaging situation, which explains physically the origin of nonuniversal exponents.

3. DISCUSSION

Usually one says a system is self-averaging if one can divide it into subsystems which are representative samples of the whole. The size of the subsystems is determined by the distance over which correlations persist. When such correlations persist to all scales, the system is no longer self-averaging. In this work we have characterized how the correlations diverge for random resistor networks as a function of the disorder strength, as measured by the broadness of the MPD. The limit in which correlations diverge depends on whether the geometrical support is critical or not. While for lattices above the threshold only exponentially broad distributions induce diverging correlations, lattices at the threshold show infinite correlation at a critical broadness given by μ_c . The disorder length ξ_D discussed in the previous sections is a measure of such correlations. The study of this length scale has shown that its critical behavior is analogous to that of ordinary geometrical percolation given that $p \leftrightarrow \mu$. The length scale ξ_D is in fact well known qualitatively in many contexts: In the study of porous media the *porosity* does not acquire its intensive macroscopic value until one measures beyond a given length scale. This effect is expected to influence the scaling of other transport-related quantities,^(21,22) such as the absolute and relative permeability, of crucial relevance in oil recovery. In the context of ore mining the length scale is known as the “nugget effect” and is easily measurable.⁽²³⁾ The knowledge of this length is very important to the scaling theories of these systems, generally described in terms of mean-field-type approaches.⁽²⁴⁾ One of the most important drawbacks of the mean field descriptions is the problem of making correct assessment of the fluctuation correlation length. On the other hand, if by an alternate theory one can find the correct correlation length, one could then incorporate such information in the original mean field scheme to improve results. One way to do this is to obtain the renormalized distribution beyond the crossover length (or merely its mean and variance) and use it as the MPD in the mean field self-consistent equation. This procedure has been carried out by Monte Carlo on regular lattices⁽²⁵⁾ with good results.

In the following we will briefly discuss other disordered systems that exhibit analogous features to RRN with a broad distribution of bond strengths. The simplest disordered system with nontrivial properties may be a random walker, whose displacement is given by the addition of random variables. The limit PDF of a sum of random variables is well known to

obey the central limit theorem always that the first two moments of the added variables exist. When the distribution of the individual variables is broad enough their first and second moments diverge and the Gaussian basin is not approached. In its place a Levy basin arises in which the limit distributions preserve the tails of the underlying random variables. In the language of Eq. (10), $\mu_c = 2$ is the critical value of the tails that separate the Gaussian basin from the Levy basin. As implied by our results, there is a strong parallelism between limit theorems for addition of random variables and RRN for lattices at p_c (see Section 2) where $\mu_c = 0.75$, or lattices above p_c with $\mu_c = 0$. Derrida⁽²⁶⁾ has established a similar analogy to spin glasses. In that model the high-temperature phase is characterized by a peaked distribution (annealed limit) well described by its average, while the low-temperature phase (spin-glass phase) corresponds to a broad distribution of properties. For the random energy model⁽²⁷⁾ (the simplest spin-glass) the correspondence $\mu \leftrightarrow T/T_c$ can be established,⁽²⁸⁾ where T_c is the critical temperature for the transition. Presumably the same will be true for the more complicated Sherrington–Kirkpatrick model.⁽²⁹⁾

Furthermore, a similar phenomenon occurs in the context of directed polymers⁽³⁰⁾ where the composition law for random variables involves sums of products. Recently Zhang⁽³¹⁾ found that nonuniversal exponents occur in the context of *directed polymers* if one introduces a broad distribution of random variables. This type of disorder presumably arises, in the context of evolving fluid interfaces, due to the power law nature of the breaking process that generates a porous medium. This model was also studied by Roux *et al.*⁽¹²⁾ and the results are very similar to ours, namely, the existence of a percolating limit (where AHL-type arguments apply well) as the broadness of the distribution increases toward exponential, and an analog of a “Levy basin.” Roux *et al.* are also able to find the asymptotic distributions and scaling behavior for the simple limit of the *optimal paths*. It would be interesting to study similar “disorder lengths” and their divergence in these closely related systems.

As a final point we discuss the action of the *composition law* for random variables as a “filter” for broad tails. From the results of the previous sections the stable limit PDF can be inferred (by inspection) from the knowledge of the microscopic PDF and the manner in which the composition law preserves, reduces, or enhances the tail of a probability distribution. We will illustrate this point by discussing the simple model of invasion percolation. In the context of this model, the interplay between composition law and PDF yields an explanation for the percolating backbone nature of the invading fluid. According to the rules of invasion percolation a nonwetting fluid will invade a porous medium characterized by a distribution of pore radii $Q(r)$ [or equivalently by a distribution of capillary

pressure $K(P)$, where $r \sim 1/P$], invading one pore at a time and following the path of least capillary resistance. For a hierarchical lattice such as the one shown in Fig. 1 the composition law to produce the equivalent capillary pressure of a cell is

$$P(\text{Equivalent}) = \max[\min(P_1, P_2), \min(P_3, P_4)] \quad (12)$$

The P_i stand for the capillary pressures assigned to the bonds of the hierarchical lattice. Such a composition law will progressively maximize the tail of any microscopic PDF, regardless of whether it is broad or not. In fact the former composition law is identical to the one of conductances in the limit that MPD is $P(g) \sim 1/g$.⁽²²⁾ In view of the previous conclusion, the AHL criterion holds and the large-scale equivalent capillary pressure of the porous medium above the percolation threshold is identical to that of the percolating cluster of smallest capillary pressure. The same argument applies for wetting fluids replacing $\min \rightarrow \max$ and vice versa in Eq. (12).

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